BALLISTICS AND DYNAMICS OF MISSILES

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The fields of Missile Ballistics and Missile Dynamics are so closely related to each other that one has to define both fields carefully, if one wants to draw a dividing line between them. Both disciplines deal with the motion of missiles in space. (The word space is used here in the sense customary in mathematics, not as space travelers use it. This means our space may be filled by air of any density, including zero density.)

Dynamics, meaning motion caused by acting forces, could include any motion of missiles. Ballistics, derived from the Greek word βάλλειν, to throw, could also be applied to any phenomenon connected with the hurling of a missile through space. Sticking to the established meaning of these words we define:

[illegible] Missile Ballistics is the theory of the trajectories of [illegible] of gravity of missiles. - Data of missile ballistics [illegible] served by theodolites.

[illegible] Missile Dynamics is the theory of the fine-structure of [illegible] trajectories including rotary motion of the missile about its [illegible] of gravity and elastic deformations. -As a rule, data of [illegible] dynamics can be observed only by instruments installed [illegible] the missile.

Security regulations have separated the science of ballistics from other sciences a long time before airplanes and missiles even existed. Therefore ballisticians had to operate their own test facilities and did not see any need for adapting their terminology and symbols to neighbouring fields, aerodynamics, for instance, when these fields were built up. We will disregard this fact and will use the terminology of conventional aerodynamics, where they apply.

No classified information will be given in this course.

Ballistics in general is divided in three fields:

Interior Ballistics
Exterior Ballistics
End Ballistics

Interior Ballistics deals with the phenomena which occur inside the gun barrel, which accelerates the projectile. End Ballistics treats the effect of the projectile on its surroundings at the end of the trajectory. This course is restricted to the Exterior Ballistics of Missiles.

Exterior ballistics of projectiles usually includes the aerodynamics of the shell, as for instance, drag problems and the selection of an optimum shape of the shell. In missile ballistics these problems are treated under the heading of missile aerodynamics.

A remark has to be made on BALLISTIC TRAJECTORIES, which are defined as trajectories without aerodynamic sideforces, or since missiles with ballistic trajectories usually have two planes of symmetry, with zero angle of attack and zero angle of sideslip. Our Missile Ballistics will not be limited to this kind of trajectories. If one looks closely into the forces acting on this type of trajectories, one would find that all types of trajectories which have curvature or are subject to natural wind, have lift forces. This is demonstrated by the sketch on the side. The angle of attack varies along the longitudinal axis of the missile, because the angular velocity, ω_y, about the lateral axis causes a local change of the angle of attack. Consequently there is a pressure distribution along the missile, which has a resultant side-force, which is different from zero.

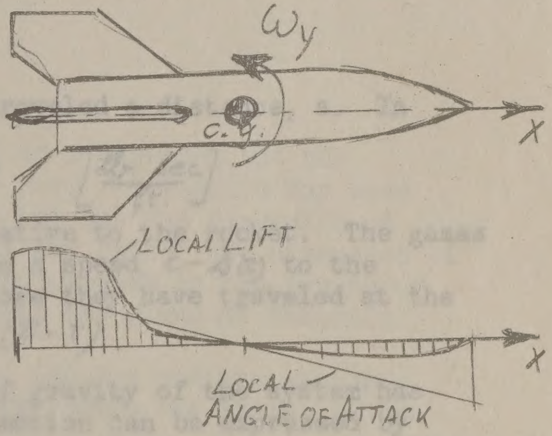


Fig. 1. The angular velocity about the lateral axis produces a lift force.

THE ROCKET EQUATION

A rocket is a reaction type engine which produces thrust, T, by ejecting gases of burned propellant with an ejection speed, c. The rocket contains all ingredients necessary for combustion in its propellant and is in every respect independent from the presence of air.

We derive the equation of motion of a rocket in a space free from air and gravity from the principle that in this type of space the center of gravity of a system retains its location in space as long as internal forces act only. If one considers the expelled gases part of the system, the rocket thrust is an internal force.

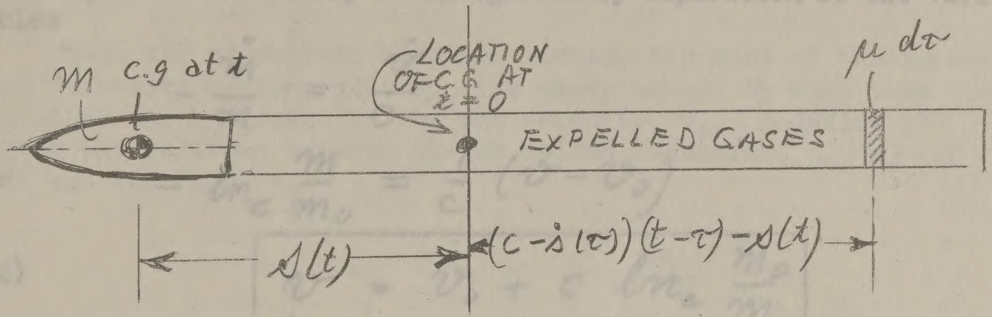


Fig. 2. Snapshot of a rocket and its trail of expelled gases at the time, t.

At the time, t, the rocket has traveled a distance, s. In each second the mass

$$\mu = -\dot{m} \quad \left[\frac{lb\ sec}{ft}\right]$$

has been ejected with a speed, c, relative to the rocket. The gases which were ejected at time, $t = \tau$, have a speed $c - \dot{s}(\tau)$ to the right relative to the rocket. Therefore they have traveled at the time, t, a distance $d = (c - \dot{s}(\tau))(t - \tau)$

The condition, that the center of gravity of the system has not moved since the beginning of the motion can be expressed by the statement: the moment of the masses about the center of gravity location at the beginning of the motion is zero. This is expressed in a formula:

$$m(t) \cdot s(t) = -\int_0^t \dot{m}\, d\tau \left[(c - \dot{s}(\tau))(t - \tau) - s(\tau) \right]$$

Differentiation with respect to t gives, if one considers that $\dot{m}$ is a constant and that the variable, t, is also contained in the upper limit of the integral:

$$\cancel{\dot{m}} s + m \dot{s} = \cancel{\dot{m}} s - \int_0^t \dot{m}\, d\tau (c - \dot{s}(\tau))$$

A second differentiation with respect to t yields:

$$\cancel{\dot{m}} \dot{s} + m \ddot{s} = -\dot{m} c + \cancel{\dot{m}} \dot{s}$$

If one writes for s the variable v, one obtains the rocket equation

(1) $$\boxed{m\dot{v} = -c\dot{m}}$$

This equation can easily be integrated by separation of the variables

$$-\frac{\dot{m}}{m} = \frac{\dot{v}}{c}$$

or $$-\ln_e \frac{m}{m_0} = \frac{1}{c}(v - v_0)$$

(2) $$\boxed{v = v_0 + c \ln_e \frac{m_0}{m}}$$

or

(3) $$\frac{m_0}{m} = e^{\frac{v-v_0}{c}}$$

Usually the exhaust velocity, c, is not given, but the specific impulse, I. The name of the specific impulse may need an explanation. The specific impulse has not the dimension of an impulse, which would be [lb · sec] but rather the dimension of time. The physical meaning of specific impulse is the length of time one pound of propellant can produce one pound of thrust.

The relation between the specific impulse and the exhaust velocity, c, can be obtained, if one expresses the thrust in two ways, by means of c and by means of I.

The only force acting on the rocket under consideration is the thrust. According to Newton's law of motion is

(mass) · (acceleration) = acting forces.

Consequently, the term $-c\dot{m}$ in equation (1) must be the thrust, T. The impulse produced by the thrust in the time, Δt, is $T \cdot \Delta t$. During this time the weight of the propellant has changed by $-\dot{m}g\,\Delta t$, which contained the impulse $-I\dot{m}g\,\Delta t$. Both impulses are equal and one obtains by comparison of both expressions:

(4) $$\boxed{c = I \cdot g}$$

g, here, is the constant of gravitation, which is on the earth approximately

(5) $$g = 32.2 \quad ft/sec^2$$

When all propellant has been burned, the mass of the rocket has decreased to the value m_e, the empty mass. By that time the velocity has increased to the end velocity, v_e. Equation (2) gives the relation between the so called Massratio, m_o/m_e and the end velocity, v_e:

(6) $$v_e = v_o + c \ln_e \frac{m_o}{m_e}$$

If one instroduces the mass of the propellant, m_p, one has

$$m_o = m_e + m_p$$

and instead of (6)

(6a) $$v_e = v_o + c \ln_e \left(1 + \frac{m_p}{m_e}\right)$$

If one has a two-stage rocket, the end speed of the first stage, $(v_e)_1$, becomes the initial velocity, $(v_o)_2$ of the second stage. One obtains:

(7) $$v_{e_2} = v_{o_1} + c \ln_e \left[\left(\frac{m_o}{m_e}\right)_1 \times \left(\frac{m_o}{m_e}\right)_2\right]$$

The product $(m_o/m_e)_1 \times (m_o/m_e)_2$ is called the compound mass ratio of the two stage rocket.

It is obvious, how equation (7) has to be generalized, if one wants to use it for a rocket of more than two stages and for configurations, which have different exhaust velocities in the different stages.

TRAJECTORIES WITHOUT AIR FORCES AND WITHOUT THRUST

The equations of the trajectories contain a number of experimental functions and can therefore be solved numerically only. Before the development of the electronic computers, the ballistic offices employed a great number of people who had to operate desk computers in order to produce the necessary numerical solutions. It is obvious that in a course on missile ballistics of only 150 minutes, there is no time for discussing the many different approaches for the numerical solution of the trajectory equations. On the other hand, it is the purpose of this course to give you a bird's eye view of the different methods used in missile ballistics and to provide you with the tools which enable you to solve some practical problems and to understand the more involved problems. For these reasons we start out here with the discussion of the flight of a ballistic missile in empty space after burnout, which covers the longest part of the trajectory of a ballistic long range missile. In the past two different methods have been developed and used for the calculation of this type of trajectory.

(a) The original approach was the so called CORRECTION METHOD. The trajectory is first calculated under the assumption that the gravity force is constant with respect to magnitude and direction. The resulting trajectory is then a parabola. Afterwards corrections are applied which take care of the variation of the gravity force.

(b) The ORBIT METHOD treats the missile like a satellite flying in the gravitational field of the earth.

It is obvious that the corrections on the parabola trajectory become the larger the longer the range of the missile. Experience has shown that the second method works faster than the first one as soon as the range of the missile exceeds some hundred miles. Therefore we will discuss the ORBIT METHOD only.

We shall approach the discussion of this method in two steps: First it is assumed that the earth does not rotate, later the rotation of the earth is considered.

ORBIT FORMULAE

It is assumed that the part of the trajectory we deal with here begins in a point, B, which is located high enough above sea level so that one is allowed to neglect aerodynamic forces in this altitude. The velocity in point B may be given according to magnitude and direction.

Kepler's first law states that the trajectory is an ellipse (Ref. 9) with the center of the earth located in that focus of the ellipse, which is further away from the trajectory part of the ellipse. (See Figure 3) The rotation of the earth has no effect on the form of the trajectory because there is no connection with the earth after the missile has left the atmosphere. The only effect caused by the rotation of the earth is that one would measure from the rotating earth a different initial velocity. This correction is given later in explicit form.

All symbols used in the following formulae are defined in Fig. 3 and Fig. 4. It should be noted that the beginning, B, and the end, T, of the part of the trajectory, for which the following formulae are valid, have the same altitude above ground.

The trajectory equation in polar coordinates is

$$r = \frac{v_{h1}^2}{g_1(1-\varepsilon \cos \varphi)} \tag{8}$$

g_1 is the gravitational acceleration in the initial point, B. It should be noted that the gravitational acceleration decreases according to the formula

$$g = g_0 \frac{r_0^2}{r^2} \tag{9}$$

where r_0 is the radius of the earth and g_0 the gravitational acceleration at sea level.

Introducing the abbreviation

$$\delta = v_{r1}/v_{h1} \tag{10}$$

for the ratio of the components of the velocity at B and the geodetic parameter

$$k = v_{h1}^2 / r_1 g_1 \tag{11}$$

the numerical eccentricity, ε , of the trajectory ellipse can be written

(12) $$\varepsilon^2 = (1 - k)^2 + \delta^2 k^2$$

and the half angles, φ_1 , of the trajectory arc of the ellipse

(13) $$\operatorname{tg} \varphi_1 = \frac{\delta k}{1 - k}$$

Equation (13) gives the range, R, of the trajectory

(14) $$R = 2 \varphi_1 r_1$$

The radial and the horizontal velocity, in a point with the centri angle, φ , are respectively

(15) $$v_r = dr/dt = -v_{h1} \frac{\varepsilon}{k} \sin \varphi$$

(16) $$v_h = r \frac{d\varphi}{dt} = v_{h1} \frac{r1}{r} = v_{h1} \frac{1 - \varepsilon \cos \varphi}{k}$$

If one introduces in (8) the geodetic parameter, k, one has

(17) $$r = r_1 \frac{k}{1 - \varepsilon \cos \varphi}$$

and the peak altitude for $\varphi = 0$ becomes

(18) $$r_{max} - r_1 = r_1 \left(\frac{k}{1 - \varepsilon} - 1\right)$$

The speed in the peak point is

(19) $$v_{hp} = v_{h1} \frac{1 - \varepsilon}{k}$$

The flight time from the initial point, B, to the target point, T, is

(20) $$t_f = \frac{2v_{r1}}{g_1} \frac{2k}{\delta(1-\varepsilon^2)} \left[\frac{\tan^{-1}\left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon} \tan \frac{\varphi_1}{2}\right)}{\sqrt{1-\varepsilon^2}} + \frac{\varepsilon \tan \frac{\varphi_1}{2}}{1-\varepsilon+(1+\varepsilon)\tan^2\frac{\varphi_1}{2}} \right]$$

SOLUTION OF THE TWO BASIC BALLISTIC PROBLEMS

We introduce spherical coordinates on the sphere, which is concentric with the earth and contains the initial point, B, and the target, T, (See Figure 4).

ϕ = geographical latitude

Θ = geographical longitude, zero for the meridian of the initial point, B.

ψ = angle defining the direction of the flight path in the initial point.

The FIRST BASIC BALLISTIC PROBLEM can then be formulated as follows:

Given are the coordinates r_1, ϕ_1, and $\Theta_1 = 0$ of the initial point, B, the velocity components v_{r1} and v_{h1} in point B, and the heading direction, ψ, of the speed v_1. Wanted are the coordinates $\phi_2 = \phi_1 + \Delta\phi$ and $\Theta_2 = \Delta\Theta$ of the target point, T.

The solution for not rotating earth is given by the orbit formulae. One calculates the parameters σ and κ by means of equations (10) and (11) and obtains the range angle $2\varphi_1$, from (13). The numberical eccentricity follows from (12) and the flight time, t_f, from (20).

The coordinates of the target point, T, can be derived by simple spherical trigonometry from the triangle BTN in Figure 4. One has the relations

$$(21) \qquad \sin(\phi_1 + \Delta\phi) = \sin\phi_1 \cos 2\varphi_1 + \cos\phi_1 , \sin 2\varphi_1 \sin\psi$$

$$(22) \qquad \sin\Delta\Theta = \frac{\sin 2\varphi_1 \cos\psi}{\cos(\phi_1 + \Delta\phi)}$$

These are the formulae for not rotating earth, where the velocity components v_{r1} and v_{h1} measured from the earth are the same as in inertial space. In the following the effect of the rotation of the earth is considered and the velocity components v_{r1} and v_{h1}, which are now measured from the rotating earth, are converted into the components v_{r_0} and v_{h_0} relative to the not rotating space. This conversion is obtained by considering the circumferential speed

$$(23) \qquad u = r_1 \omega \cos\phi$$

[illegible]

[illegible] can then be formulated as [illegible]

[illegible] at the target point [illegible]

[illegible]

[illegible] cos [illegible] cos [illegible]

[illegible]

of the earth in the initial point, B. ω is the angular velocity of the earth, which has the numerical value $\omega = 0.72905 \quad 10^{-4}$ rad/sec. One obtains

$$v_{r_o} = v_{r_1} \tag{24}$$

$$v_{h_o} = v_{h_1} \sqrt{1 + 2\mu \cos\psi + \mu^2} \tag{25}$$

where

$$\mu = \frac{u}{v_{h_1}} = \frac{r_1 \omega \cos\phi}{v_{h_1}} \tag{26}$$

The direction, ψ, has also to be corrected for earth rotation and converted into ψ_o, which may be determined by either of the following formulae

$$\sin\psi_o = \frac{\sin\psi}{\sqrt{1 + 2\mu \cos\psi + \mu^2}} \tag{27}$$

$$\cos\psi_o = \frac{\mu + \cos\psi}{\sqrt{1 + 2\mu \cos\psi + \mu^2}} \tag{28}$$

Likewise the flight parameters δ and k are affected by earth rotation. They have to be changed into

$$\delta_o = \frac{\delta}{\sqrt{1 + 2\mu \cos\psi + \mu^2}} \tag{29}$$

$$k_o = k\,(1 + 2\mu \cos\psi + \mu^2) \tag{30}$$

Substituting (29) and (30) into (12), (13), and (20) for δ and k, one obtains the eccentricity, ε_o, the range angle, $2\varphi_o$, and the flight time, t_{f_o}, of the trajectory in the nonrotating space. (21) and (22) provide the coordinates on the sphere through B and T.

While the missile moved from B to T, the earth has rotated in underneath the trajectory by an angle $\omega \cdot t_{f_o}$. Consequently the coordinates of the actual target point, T, on the rotating earth are

(31) $$\Delta\phi = \Delta\phi_0$$

(31) $$\Delta\Theta = \Delta\Theta_0 - \omega t_{f0}$$

If one wants to know the errors caused by neglecting the rotation of the earth, one has to do it the hard way performing the calculations for rotating and for not rotating earth. There is no easier way.

FLIGHT TIME AND RANGE ERRORS OF PARABOLA TRAJECTORIES

The errors caused by simplifying the orbit trajectory so that it becomes a parabola can be given in relatively simple form for the flight time and for the range. The flight time of a parabola trajectory is $2v_{r1}/g_1$. Comparing this value with (20) one sees that the flight time is in error by the factor

$$\frac{2k}{\delta(1-\varepsilon^2)}\left[\frac{\tan^{-1}\left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon}\tan\frac{\varphi_1}{2}\right)}{\sqrt{1-\varepsilon^2}} + \frac{\varepsilon\tan\frac{\varphi_1}{2}}{1-\varepsilon+(1+\varepsilon)\tan^2\frac{\varphi_1}{2}}\right]$$

The range of a parabola trajectory is $(2v_{r1}\,v_{h1})/g_1$. If one deals with a missile for which the range angle $2\varphi_1$ is small enough so that $\tan\varphi_1$ may be replaced by φ_1 one obtains from (13)

$$R = 2\varphi_1 r_1 = \frac{2v_{r1}\,v_{h1}}{g_1(1-k)}$$

Consequently the range of the parabola is wrong by a factor $1/(1-k)$.

THE SECOND BASIC PROBLEM OF BALLISTICS

The basic ballistic problem occurring in practice is an inversion of the first ballistic problem discussed so far. It is called the second ballistic problem and can be formulated as follows:

The coordinates of the initial point of the trajectory and the coordinates of the target point are given. This means one knows r_1, ϕ_1, $\Delta\varphi$, $\Delta\Theta$. Wanted are the velocity vector and its heading

in the initial point of the trajectory, this means v_{r1}, v_{h1}, and ψ have to be determined.

There is no closed solution known to this problem. One proceeds as follows. The rotation of the earth is first neglected and one calculates ψ and φ_1 from (21) and (22) and from (13) a pair of corresponding values δ and k. The necessary velocity components are then given by (10) and (11). Now, (20) gives the flight time and one can correct the results for earth rotation. After this, the calculation is repeated. In practice, one calculates large families of solutions of the FIRST BALLISTIC PROBLEM and selects from these solutions the one which has the desired initial point and target point.

OPTIMUM RANGE

It is of interest to know, under which elevation angle one has to start the trajectory, if one wants to obtain the longest possible range with a given initial speed. The answer can be derived from (13). Introducing the angle ϑ_0 between the vertical and the velocity vector and substituting in (13)

$$v_{h_1} = v_0 \sin \vartheta_0$$

$$v_{r_1} = v_0 \cos \vartheta_0$$

and differentiating with respect to ϑ_0 one obtains for optimum ϑ_0 the condition

$$\text{(32)} \qquad \cos 2\vartheta_0 = - \frac{1}{2\frac{r_1 g_1}{v_0^2} - 1} \approx - \frac{v_0^2}{2 r_1 g_1}$$

For missiles, for which $\frac{v_0^2}{2r_1g_1}$ is small the optimum elevation angle of the vector of the initial speed, $90^\circ - \vartheta_0$, is only somewhat smaller than 45°. For missiles with longer range the optimum elevation angle can become appreciably smaller than 45°.

NUMERICAL EXAMPLE

The formulae may be illustrated by some numbers taken from Ref. 7. The example is a certain variation of the V-2 missile.

Initial point is the burnout point in 30 km altitude, which gives

$$r_1 = 6{,}400 \text{ km}$$

components of burnout velocity

vertical horizon meter

$$v_{r_1} = v_{h1} = 1{,}000 \text{ m/sec}$$

$$g_1 = 9.6963 \text{ m/sec}^2$$

$$\Phi_1 = 50^\circ$$

For a parabolic trajectory is the range

$$R = (2v_{r_1} v_{h1})/g_1 = 206.264 \text{ km}$$

The side deviation depends upon the flight direction, which was symbolized by ψ. It lies between 3.64% and -1.3%, the range deviations between 2.81% and 0.24%.

The flight time is for parabolic trajectory

parabolic $$t_{f_p} = 2v_{r1}/g_1 = 206.264 \text{ sec} \quad \text{(same number as range because } v_{h1} = 1{,}000\text{)}$$

Without rotation of the earth the flight time is for orbit trajectory

eleptic $$t_{fe} = 211.935 \text{ sec}$$

If the rotation of the earth is considered, the flight time depends upon the direction of flight and varies between 210.170 and 214.363 sec.

CORRECTIONS FOR AIR DRAG

For the ascent part in dense air (from the ground to point B in Figure 3) the influence of air drag and lift has to be taken care of in detail as will be seen later. For the reentry part of the trajectory (from T to the ground) the same is true. There exist, however, cases where the burnout point, B, is not quite high enough so that one would not like to neglect the air drag completely (though an exact calculation with drag might be considered unnecessary). For such cases the following formulae are useful.

Range correction

$$(33) \qquad \Delta R = - \frac{S\, c_D}{2m} \rho_0 \int_0^{t_f}\int_0^{t_f} \frac{\rho}{\rho_0} v v_h \, dt\, dt$$

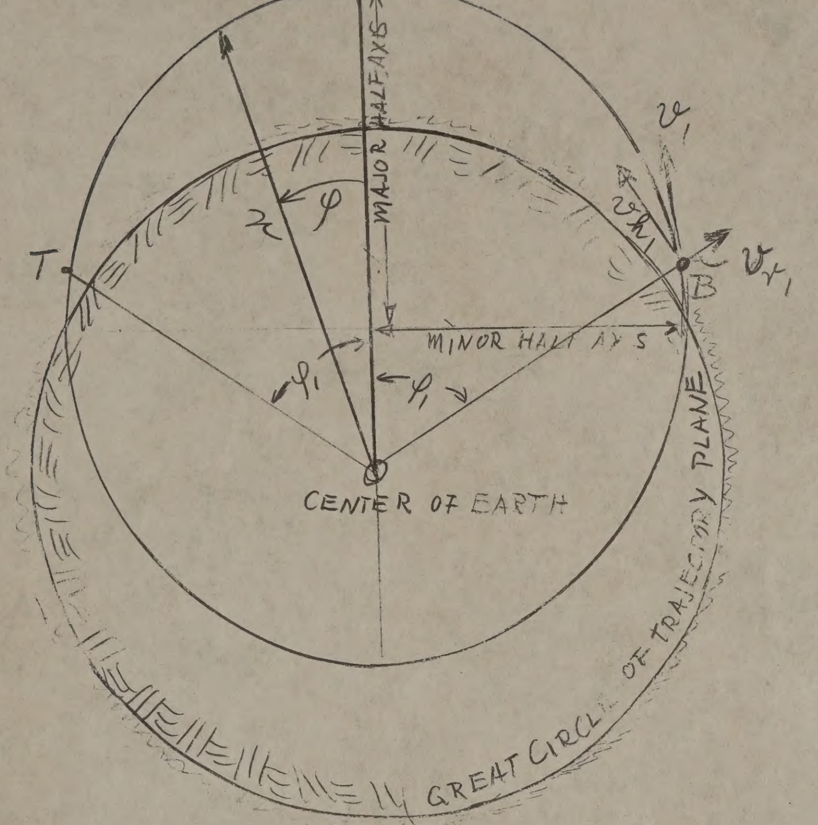
and for the sideward deviations caused by a wind velocity, w, perpendicular to the missile velocity, v,

$$(34) \qquad \Delta S = - \frac{S\, c_D}{2m} \rho_0 \int_0^{t_f}\int_0^{t_f} \frac{\rho}{\rho_0} w v \, dt\, dt$$

Since the trajectory is known, the integrals (33) and (34) can easily be evaluated. In many cases it may be sufficient to use an average speed, v_m, between burnout and the peak and to use for the variable horizontal speed, v_h, the horizontal speed at burnout, v_{h_1}. Then the equations are simplified as (35) and (36);

$$(35) \qquad \Delta R = - \frac{S'\, c_D}{2m} \rho_0 v_m v_{h_1} \int_0^{t_f}\int_0^{t_f} \frac{\rho}{\rho_0} \, dt\, dt$$

$$(36) \qquad \Delta S = - \frac{S\, c_D}{2m} \rho_0 v_m^2 \int_0^{t_f}\int_0^{t_f} \frac{\rho}{\rho_0} \frac{w}{v_m} \, dt\, dt$$



$R = \frac{v_h^2}{g\cdot(1-2\cos\text{[illegible]})}$

Fig. 3. DEFINITION OF SYMBOLS FOR ORBIT TRAJECTORY

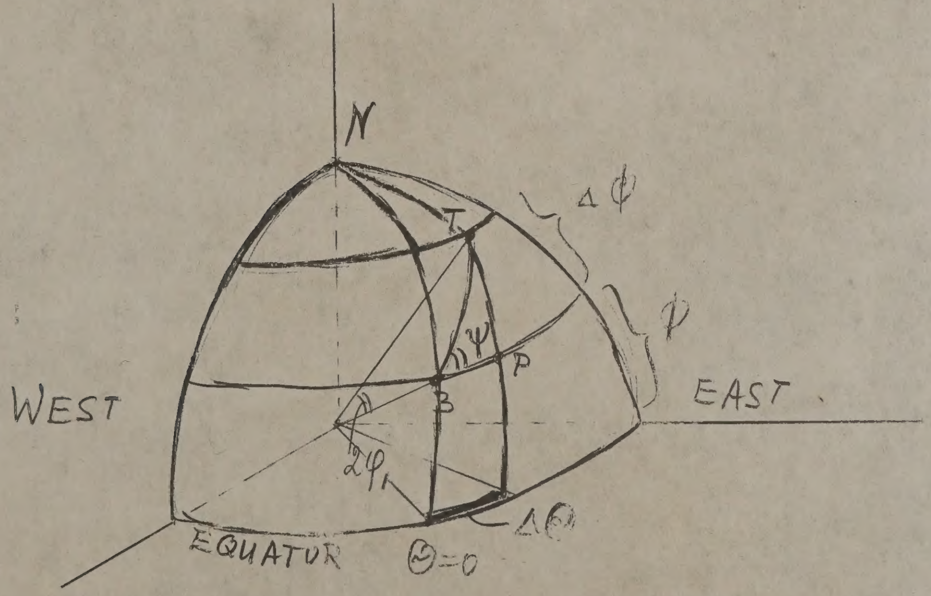


Fig. 4. COORDINATES ON THE SPHERE THROUGH BURNOUT POINT B AND TARGET POINT T